3.3: Raonal Funcons of the Form $f(x)=\frac{ax+b}{cx+d}$

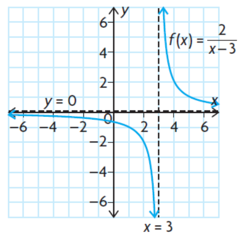
Key Ideas:

$f(x)=\frac{p(x)}{q(x)}$

- Raonal funcons of the form $\frac{b}{cx+d}$ have a vercal asymptote defined by $x=\frac{-d}{c}$ and a horizontal asymptote defined by $y=0$. For example, the graph of $f(x)=\frac{2}{x-3}$

$cx+d=0$

$x=\frac{-d}{c}$

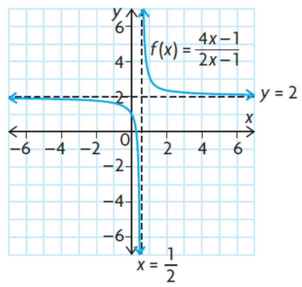


- Most raonal funcons of the form $f(x)=\frac{ax+b}{cx+d}$ have a vercal asymptote defined by $x=\frac{-d}{c}$ and a horizontal asymptote defined by $y=\frac{a}{c}$. For example see the graph of $f(x)=\frac{4x-1}{2x-1}$

$4/2 = 2$

$$f(x)=\frac{\frac{ax}{x}+\frac{b}{x}}{\frac{cx}{x}+\frac{d}{x}}$$

$$=\frac{a+\frac{b}{x}\to 0}{c+\frac{d}{x}\to 0}$$

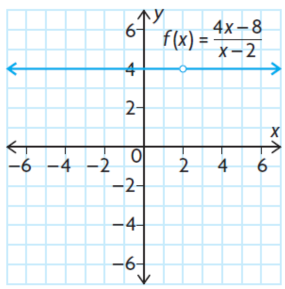


- The excepon occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factors occurs. As a result, the graph has no asymptotes. For example see the graph of $f(x)=\frac{4x-8}{x-2}$

$$f(x)=\frac{4(x-2)}{(x-2)} \quad x\neq 2$$

$y=4$

$$f(a)=\frac{p(a)}{q(a)}=\frac{0}{0}$$



Exploring Quoents of Polynomial Funcons

Note:

- The quoent of two polynomial funcons results in a raonal funcon which oen has one or more disconnuies.
- The breaks or disconnuies in a raonal funcon occur where the funcon is undefined. The funcon is undefined at values where the denominator is equal to zero. As a result, these values must be restricted from the domain of the funcon.
- The values that must be restricted from the domain of a raonal funcon result in key characteriscs that define the shape of the graph. These characteriscs include a combinaon of vercal asymptotes (also called infinite disconnuies) and holes (also called point disconnuies).
- The end behaviour of many raonal funcons is determined by either horizontal asymptotes or oblique asymptotes.
- An oblique asymptote is an asymptote that is not vercal nor horizontal, but slanted.

Point Disconnuies (Holes)

Hole Disconnuies

- A raonal funcon, $f(x)=\frac{p(x)}{q(x)}$ has a hole at *x = a,* if $\frac{p(a)}{q(a)}=\frac{0}{0}$. This occurs when *p(x)* and *q(x)* contain a common factor of *(x - a)*.

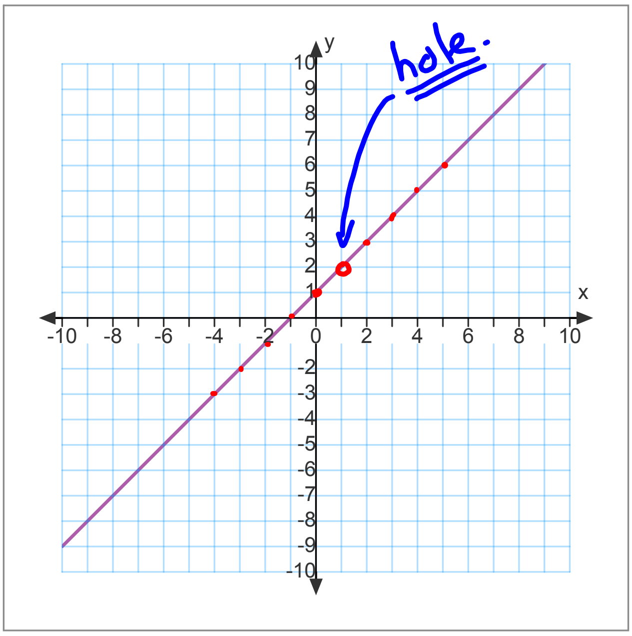
- Consider the following raonal funcon:

$$y=\frac{x^2-1}{x-1}$$

$$f(1)=\frac{1^2-1}{1-1}=\frac{0}{0}$$

$$y=\frac{(x-1)(x+1)}{(x-1)}$$

$$y=x+1$$



Point Disconnuies (Holes)

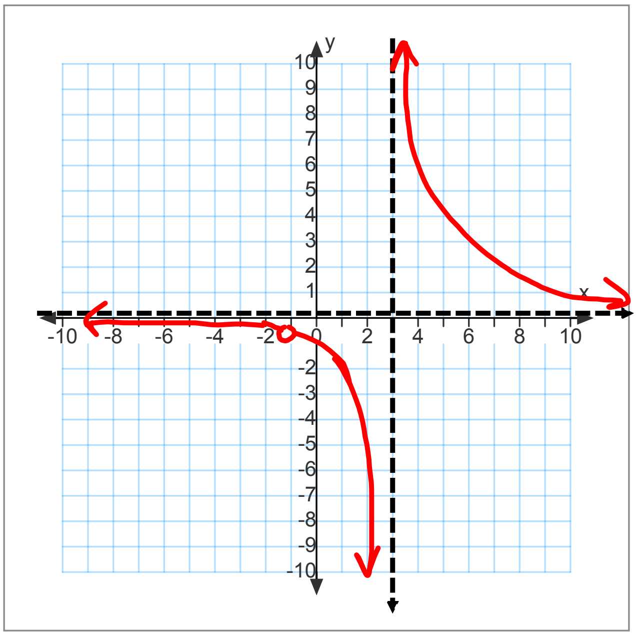
- Consider the following raonal funcon:

$$y = \frac{x+1}{x^2 - 2x - 3}$$

$$f(-1) = \frac{0}{1+2-3} = \frac{0}{0}$$

$$y = \frac{x+1}{(x-3)(x+1)}$$

$$y = \frac{1}{(x-3)}$$



Check for H.A:

$$= \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{3}{x^2}}$$

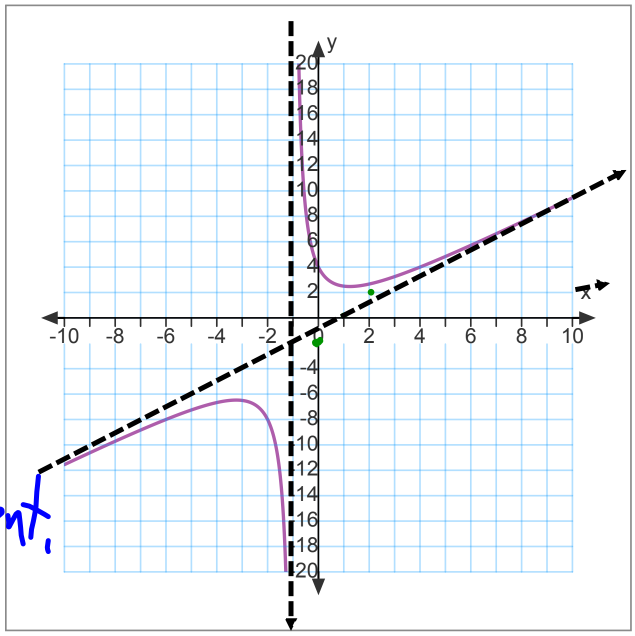
$$= \frac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{2}{x} - \frac{3}{x^2}} = \frac{0}{1} = 0$$

hole at $x = -1$

Horizontal asymptote at $y = 0$

Oblique (slant) Asymptotes

- Consider the following raonal funcon:



$$y = \frac{x^2+4}{x+1} \rightarrow x \neq -1$$

to find the eqn of the slant asymptote, divide $p(x) \div q(x)$ find the quotient.

$p(x) = x^2+4$

$q(x) = x+1$

$p(x) \div q(x) =$

$$\begin{array}{r} x - 1 \\ x+1 \overline{) x^2 + 0x + 4} \\ \underline{x^2 + 1x \quad\ \ } \\ -1x + 4 \\ \underline{-1x - 1} \\ 5R \end{array}$$

∴ eqn of the slant asymptote is $y = x - 1$

Oblique (slant) Asymptotes

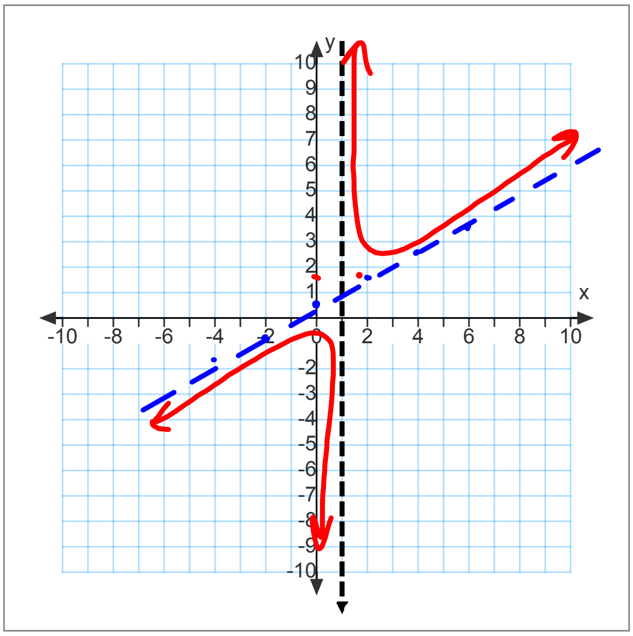
- Consider the following raonal funcon:

$$y = \frac{0.5x^2 + 1}{x - 1}$$

$y = \frac{1}{2}x + \frac{1}{2}$

$x \neq 1$

$$\begin{array}{r} 0.5x + 0.5 \\ x-1 \overline{)\, 0.5x^2 + 0x + 1} \\ \underline{0.5x^2 - 0.5x} \downarrow \\ 0.5x + 1 \\ \underline{0.5x - 0.5} \end{array}$$

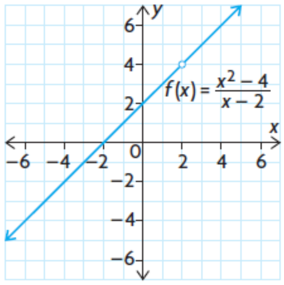


Oblique (slant) Asymptotes

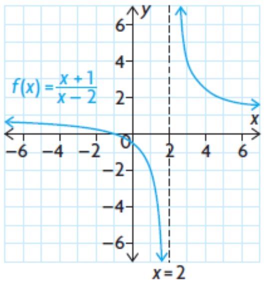
- A raonal funcon, $f(x)=\dfrac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of *p(x)* is greater than the degree of *q(x)* by exactly 1.
- To determine the equaon of the oblique asymptote, simply carry out the polynomial division $p(x)\div q(x)$. The quoent will represent the equaon of the oblique asymptote.

Key Ideas

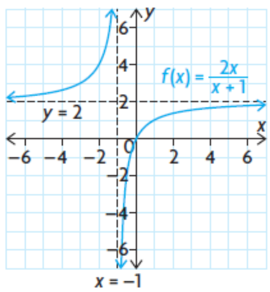
- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a hole at $x = a$ if $\frac{p(a)}{q(a)} = \frac{0}{0}$. This occurs when $p(x)$ and $q(x)$ contain a common factor of $(x - a)$.
  For example, $f(x) = \frac{x^2 - 4}{x - 2}$ has the common factor of $(x - 2)$ in the numerator and the denominator. This results in a hole in the graph of $f(x)$ at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a vertical asymptote at $x = a$ if $\frac{p(a)}{q(a)} = \frac{p(a)}{0}$.
  For example, $f(x) = \frac{x + 1}{x - 2}$ has a vertical asymptote at $x = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has a horizontal asymptote only when the degree of $p(x)$ is less than or equal to the degree of $q(x)$. For example, $f(x) = \frac{2x}{x + 1}$ has a horizontal asymptote at $y = 2$.



- A rational function, $f(x) = \frac{p(x)}{q(x)}$, has an oblique (slant) asymptote only when the degree of $p(x)$ is greater than the degree of $q(x)$ by exactly 1. For example, $f(x) = \frac{x^2 + 4}{x + 1}$ has an oblique asymptote.