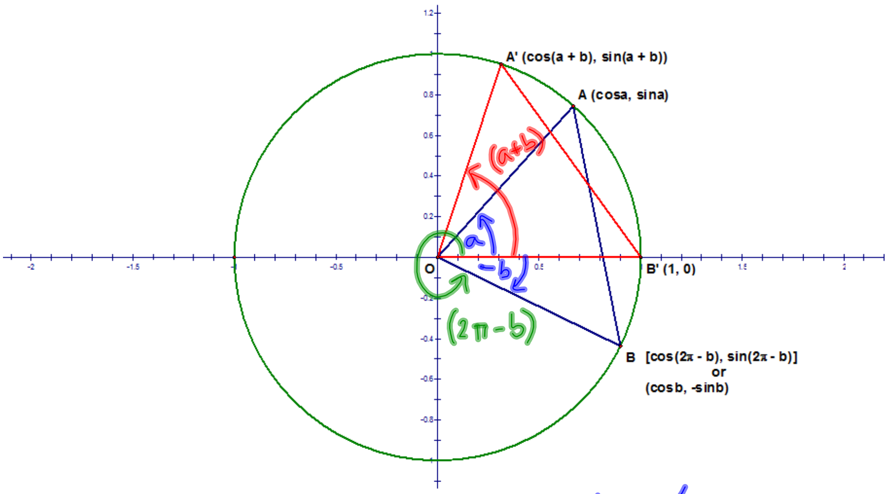
4.4: Compound Angle Formulas

- In the pages that follow, compound angle formulas will be developed using algebra and the unit circle
- The compound angle formulas for sine, cosine and tangent are:
  - > $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
  - > $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
  - > $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
  - > $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
  - > $\tan(x + y) = \tan(x) + \tan(y) / 1 - \tan(x)\tan(y)$
  - > $\tan(x - y) = \tan(x) - \tan(y) / 1 + \tan(x)\tan(y)$
- By using compound angle formulas, you are able to determine exact values for trigonometric raos that can be expressed as the sum or difference of special angles.

Proving the sum formula for cosine: cos(x+y)



A rotates 'b' degrees to A'

B rotates 'b' degrees to B'

lengths are preserved under rotation, so:

$$|A'B'| = |AB| \quad \text{* use: } d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\sqrt{[\cos(a+b)-1]^2+[\sin(a+b)-0]^2}=\sqrt{[\cos a-\cos b]^2+[\sin a+\sin b]^2}$$

$$[\cos(a+b)-1]^2+[\sin(a+b)]^2=[\cos a-\cos b]^2+[\sin a+\sin b]^2$$

$$\cos^2(a+b)-2\cos(a+b)+1+\sin^2(a+b)=\cos^2 a-2\cos a\cos b+\cos^2 b+\sin^2 a+2\sin a\sin b+\sin^2 b$$

$$\underbrace{\sin^2(a+b)+\cos^2(a+b)}+1-2\cos(a+b)=\cos^2 a+\sin^2 a+\sin^2 b+\cos^2 b-2\cos a\cos b+2\sin a\sin b$$

$$1+1-2\cos(a+b)=1+1-2\cos a\cos b+2\sin a\sin b$$

$$\cancel{2}-2\cos(a+b)=\cancel{2}-2\cos a\cos b+2\sin a\sin b$$

$$-2\cos(a+b)=-2\cos a\cos b+2\sin a\sin b$$

$$\frac{-2\cos(a+b)}{-2}=\frac{-2[\cos a\cos b-\sin a\sin b]}{-2}$$

$$\boxed{\cos(a+b)=\cos a\cos b-\sin a\sin b}$$

Proving the difference formula for cosine: cos(x - y)

The subtracon formula for cosine can be derived from the addion formula for cosine:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x+(-y)) = \cos x \cos(-y) - \sin x \sin(-y)$$

$\bigstar$ recall: $\cos(-y) = \cos y$

$\sin(-y) = -\sin y$

$$\therefore \cos(x-y) = \cos x \cos y - \sin x(-\sin y)$$

$$\boxed{\cos(x-y) = \cos x \cos y + \sin x \sin y}$$

Proving the sum formula for sin(x + y)

<u>*Recall:*</u> *Co-funcon identes involving (90 ^{0} - x)*

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$\sin(x+y) = ?$

Let $(x+y)$ be z

$\therefore \sin z = ?$

$\sin z = \cos(\pi/2 - z)$

re-substitute $(x+y)$ for z

$\therefore \sin(x+y) = \cos\underbrace{[\pi/2 - (x+y)]}_{\text{re-group}}$

$\sin(x+y) = \cos[(\pi/2 - x) - y]$

apply subtraction formula for cosine:

$\sin(x+y) = \cos(\pi/2 - x)\cos y + \sin(\pi/2 - x)\sin y$

apply co-function identity for $\cos(\pi/2 - x)$ and $\sin(\pi/2 - x)$

$\therefore \boxed{\sin(x+y) = \sin x\cos y + \cos x\sin y}$

Proving the difference formula for sin(x - y)

The difference formula for sine can be derived from the sum formula for sine:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x+(-y)) = \sin x \cos(-y) + \cos x \sin(-y)$$

★ recall: $\cos(-y) = \cos y$

$\sin(-y) = -\sin y$

$$\therefore \sin(x-y) = \sin x \cos y + \cos x(-\sin y)$$

$$\boxed{\sin(x+y) = \sin x \cos y - \cos x \sin y}$$

Proving the sum formula for tan(x + y)

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

☆ divide everything by $\cos x \cos y$

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x (1) + (1) \tan y}{1 - \tan x \tan y}$$

$$\boxed{\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

Proving the difference formula for tan(x - y)

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)}$$

$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

☆ divide everything by $\cos x \cos y$

$$= \frac{\frac{\sin x \cos y}{\cos x \cos y} - \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\tan x (1) - (1) \tan y}{(1) + \tan x \tan y}$$

$$\boxed{\therefore \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}$$

__Example 1:__ Find the exact value of $\sin\left(\frac{\pi}{12}\right)$

$$\left(\frac{\pi}{12}\right) = 15^\circ \quad \therefore \quad \sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \left(\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{6}\right) - \cos\frac{\pi}{4}\sin\frac{\pi}{6}.$$

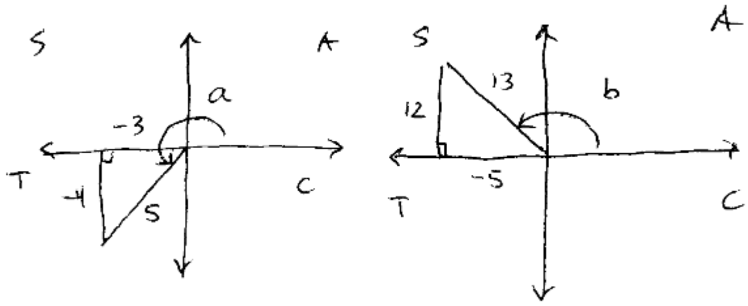
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{or} \quad \frac{\left(\sqrt{3}-1\right)}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example 2: If $\sin a = -\frac{4}{5}$, $\pi < a < \frac{3\pi}{2}$, and $\cos b = -\frac{5}{13}$, $\frac{\pi}{2} < b < \pi$, evaluate $\tan(a+b)$



$\sin a = \frac{-4}{5}$ $\qquad$ $\cos b = \frac{-5}{13}$

$\therefore \cos a = \frac{-3}{5}$ $\qquad$ $\therefore \sin b = \frac{12}{13}$

$\tan a = \frac{4}{3}$ $\qquad$ $\tan b = \frac{-12}{5}$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$= \frac{4/3 + \left(\frac{-12}{5}\right)}{1 - (4/3)(-12/5)}$$

$$= \frac{-16}{63}$$

Attachments

4.4 example 1.pdf

4.4 example 2.pdf