2.2: The Factor Theorem

Factor Theorem:

x - b is a factor of a polynomial *P(x)* if and only if *P(b) = 0*. Similarly, *ax - b* is a factor of *P(x)* if and only if $P\left(\frac{b}{a}\right)$ = 0.

Example 1

Determine if the binomials *x + 1* and *2x + 1* are factors of the polynomial $P(x) = 2x^3 + 3x^2 - 3x - 2$

$P(-1) = 2(-1)^3 + 3(-1)^2 - 3(-1) - 2$

$= -2 - 3 + 3 - 2$

$P(-1) = -4$

$\therefore$ x+1 is not a factor

$P\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right)^3 + 3\left(\frac{-1}{2}\right)^2 - 3\left(\frac{-1}{2}\right) - 2$

$P\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{8}\right) + 3\left(\frac{1}{4}\right) + \frac{3}{2} - \frac{2}{1}$

$= \frac{-2}{8} + \frac{3}{4} + \frac{3}{2} - \frac{2}{1}$

$P\left(\frac{-1}{2}\right) = 0$ $\therefore$ 2x+1 is a factor of P(x)

Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$

A value *x = b* that sasfies *P(b) = 0* also sasfies $b^3 + 2b^2 - 5b - 6 = 0$ or the equaon $b^3 + 2b^2 - 5b = 6$

$$b^3 + 2b^2 - 5b = 6$$

factor out b:

$$b\,(b^2 + 2b - 5) = 6$$

if b is an integer, then the value of $b^2 + 2b - 5$ will also be an integer.

$$\underbrace{b}_{\text{must be a factor of 6!}}\ \underbrace{(b^2 + 2b - 5)} = 6$$

The possible integer values for the factors in the product are factors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$

- The relaonship between the factors of a polynomial and the constant term in the polynomial expression is stated in the **integral zero theorem**

Integral Zero Theorem:

If $x - b$ is a factor of a polynomial $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

- Once one factor of a polynomial is found, polynomial division or synthec division is used to determine the other factors.

Example 2

Factor $x^3 + 2x^2 - 5x - 6$ fully

Find factors of 6, so that when we sub one of those #s, say x=b into P(x), we get P(b)=0

$\pm1, \pm2, \pm3, \pm6$

$P(2) = 2^3 + 2(2)^2 - 5(2) - 6$

$= 8 + 8 - 10 - 6$

$P(2) = 0$

$\therefore (x-2)$ is a factor!

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$\therefore (x-2)(x^2+4x+3)$

$\therefore (x-2)(x+1)(x+3)$

Example 3

Factor $x^4+3x^3-7x^2-27x-18$ fully.

Factors of 18:
$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$f(-3) = 0$

$f(-2) = 0 \quad \therefore \; x+2$ is a factor.

$f(1) = ?$

$f(-1) = ?$

$\cancel{f(2) = ?}$

$\boxed{f(-2) = 0}$

$$
\begin{array}{r|rrrrr}
-2 & 1 & 3 & -7 & -27 & -18 \\
 & & -2 & -2 & 18 & 18 \\
\hline
 & 1 & 1 & -9 & -9 & 0
\end{array}
$$

$(x+2)((x^3+x^2)-(9x-9))$

$(x+2)\; x^2(x+1) - 9(x+1)$

$(x+2)(x+1)(x^2-9)$

$\underline{(x+2)(x+1)(x-3)(x+3)}$

$$(x+2)(x^3+x^2-9x-9)$$

$f(3)=0$

$\therefore (x-3)$ is a factor.

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -9 & -9 \\ & & 3 & 12 & 9 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

$$(x+2)(x^2+4x+3)$$

$$(x+2)(x+3)(x+1)(x-3)$$

Consider a factorable polynomial such as $P(x)=3x^3+2x^2-7x+2$. Since the leading coefficient is 3, one of the factors must be of the form *3x - b* where *b* is a factor of the constant term 2 and $P\left(\frac{b}{3}\right)=0$

To determine the values of x that should be tested to find *b,* the integral zero theorem can be extended to include polynomials with leading coefficients that are not 1. This extension is known as the **raonal zero theorem.**

Raonal Zero Theorem:

Suppose *P(x)* is a polynomial funcon with integer coefficients and $x=\frac{b}{a}$ is a zero of *P(x),* where a and b are integers and $a \neq 0$. Then,

- *b* is a factor of the constant term of *P(x)*
- *a* is a factor of the leading coefficient of *P(x)*
- *ax - b* is a factor of *P(x)*

Example 4

The forms used to make large rectangular blocks of ice come in different dimensions such that the volume, V, in cubic cenmetres, of each block can be modelled by $V(x) = 3x^3 + 2x^2 - 7x + 2$

a) Determine possible dimensions in terms of x, in metres, that result in this volume.

(b) What are the dimensions of blocks of ice when x = 1.5?

a) b: 2, factors of 2: $\pm 1, \pm 2$.

a: 3, factors of 3: $\pm 1, \pm 3$

possibilities to test: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{1}{3}, \pm\frac{2}{3}$

$\therefore \pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$

$P\left(\frac{1}{3}\right) = 0$ $\therefore (3x-1)$ is a factor!

$$
\begin{array}{r}
x^2 + x - 2 \\
3x-1 \overline{)\, 3x^3 + 2x^2 - 7x + 2} \\
\underline{3x^3 - x^2 } \\
3x^2 - 7x \\
\underline{3x^2 - x } \\
-6x + 2 \\
\underline{-6x + 2} \\
0R
\end{array}
$$

$\therefore (3x-1)(x^2+x-2)$

$(3x-1)(x+2)(x-1)$

$\therefore$ dimensions are:

$(3x-1)$; $(x+2)$; $(x-1)$

p. 102 - 103

1- 11 (pick n choose).

12, 13, 18