2.1: The Remainder Theorem

Watch the mathmagician make the remainder disappear!

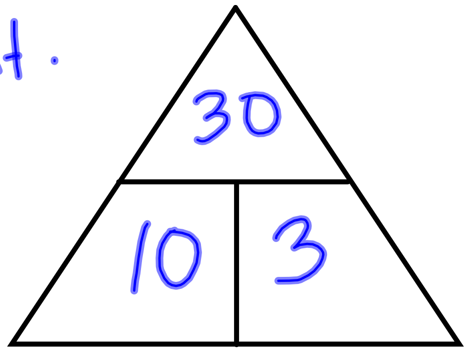
Recall: Long Division!

$$\begin{array}{r} 3 \\ 10\overline{)30} \\ \underline{30} \\ \text{OR} \end{array}$$

3 ← Quotient.

divisor

dividend

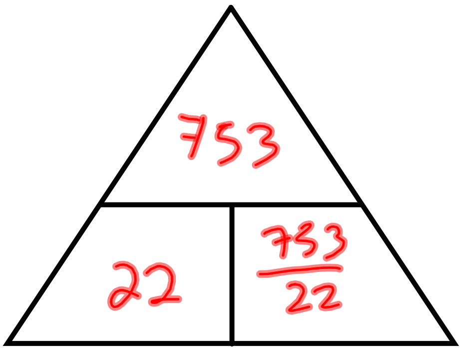


$\frac{30}{10} = 3$ $\frac{30}{3} = 10$

$10 \times 3 = 30$

Recall: Long Division with a Remainder

$$\begin{array}{r} 34 \\ 22\overline{)753} \\ \underline{66} \\ 93 \\ \underline{88} \\ 5R \end{array}$$



$$\frac{753}{22} \stackrel{?}{=} 34 + \frac{5}{22}$$

$$\frac{753}{22} \stackrel{?}{=} \frac{748}{22} + \frac{5}{22}$$

$$\frac{753}{22} \stackrel{?}{=} \frac{753}{22}$$

Quotient × divisor = dividend.

$$\frac{\text{divident}}{\text{divisor}} = \text{Quotient}$$

Polynomial Division

$$\begin{array}{r}
x+3 \\
x+2\overline{)\,x^2+5x+7} \\
\underline{x^2+2x} \\
3x+7 \\
\underline{3x+6} \\
1\ R
\end{array}$$

$$\frac{x^2+5x+7}{x+2} \overset{?}{=} x+3+\frac{1}{x+2}$$

$$\overset{?}{=} \frac{(x+3)(x+2)}{(x+2)}+\frac{1}{(x+2)}$$

$$\cancel{x+2}\left[\frac{x^2+5x+7}{\cancel{x+2}}\right] = \left[\frac{(x+3)(x+2)+1}{\cancel{x+2}}\right]\cancel{(x+2)}$$

$$x^2+5x+7 = (x+3)(x+2)+1$$

Theorem

The result of the division of a polynomial *P(x)* by a binomial of the form *x - b* is $\frac{P(x)}{x-b}=Q(x)+\frac{R}{x-b}$, where *Q(x)* is the quoent and *R* is the remainder. The corresponding statement, that can be used to check the division, is

$$\boxed{P(x)=(x-b)Q(x)+R}$$

Proof:

$$\frac{P(x)}{(x-b)}=Q(x)+\frac{R}{x-b}$$

$$=\frac{Q(x)(x-b)}{(x-b)}+\frac{R}{(x-b)}$$

$$\left(\frac{P(x)}{\cancel{(x-b)}}\right)\cancel{(x-b)}=\left[\frac{Q(x)(x-b)+R}{\cancel{(x-b)}}\right]\cancel{(x-b)}$$

$$P(x)=Q(x)(x-b)+R$$

Example 1: Divide a Polynomial by a Binomial of the Form x - b

Divide $-3x^2+2x^3+8x-12$ by $x-1$. Idenfy any restricons on the variable. Write the corresponding statement that can be used to check the division

$$
\begin{array}{r}
2x^2 - x + 7 \\
x-1\,\overline{)\,2x^3 - 3x^2 + 8x - 12} \\
2x^3 - 2x^2 \downarrow \quad\quad \\
\hline
-x^2 + 8x \quad\quad \\
\cancel{-x^2 + x} \quad\quad \\
\hline
7x - 12 \\
7x - 7 \\
\hline
-5R
\end{array}
$$

$$\frac{2x^3-3x^2+8x-12}{x-1} = 2x^2-x+7+\left(\frac{-5}{x-1}\right)$$

$x \neq 1$

To Check:

$$2x^3-3x^2+8x-12 = (x-1)(2x^2-x+7)+(-5)$$

Example 2: Divide a Polynomial by a Binomial of the Form ax - b

Divide $4x^3+9x-12$ by $2x+1$. Idenfy any restricons on the variable. Write the corresponding statement that can be used to check the division

$$\begin{array}{r} 2x^2 - x + 5 \\ 2x+1 \overline{)\, 4x^3 + 0x^2 + 9x - 12} \\ \underline{4x^3 + 2x^2 \qquad\qquad} \\ -2x^2 + 9x \qquad \\ \underline{-2x^2 - x \qquad} \\ 10x - 12 \\ \underline{10x + 5} \\ -17\ R \end{array}$$

$$\frac{4x^3+9x-12}{2x+1} = 2x^2-x+5+\left(\frac{-17}{2x+1}\right)$$

$$x \neq -\frac{1}{2}$$

To Check:

$$4x^3+9x-12 = (2x+1)(2x^2-x+5)+(-17)$$

$$\frac{25}{10} = 2 + \_\_$$

Determining a Remainder Without Dividing

Consider the binomial x - 2. Compare this binomial to the expression *x - b.* What is the value of *b*?

$b = +2$

Given a polynomial *P(x)*, what is the value of *x* in *P(2)*?

$x = +2$

Divide $P(x) = x^3 + 6x^2 + 2x - 4$ by $x - 2$. What is the remainder?

$$
\begin{array}{r|l}
 & x^2 + 8x + 18 \\
x-2 & x^3 + 6x^2 + 2x - 4 \\
 & x^3 - 2x^2 \\ \hline
 & 8x^2 + 2x \\
 & 8x^2 - 16x \\ \hline
 & 18x - 4 \\
 & 18x - 36 \\ \hline
 & 32 \; R
\end{array}
$$

$R = 32$

Evaluate *P(2).* What do you noce?

$P(x) = x^3 + 6x^2 + 2x - 4$

$P(2) = 2^3 + 6(2)^2 + 2(2) - 4$

$= 8 + 24 + 4 - 4$

$= 32$

We notice that $P(2) = 32$ which is the remainder!

Theorem: The Remainder Theorem

When a polynomial funcon *P(x)* is divided by *x - b,* the remainder is *P(b);* and when it is divided by *ax - b,* the remainder is $P\left(\frac{b}{a}\right)$, where *a* and *b* are integers, and $a \neq 0$

Proof:

Let P(x) be a polynomial divided by x-b
Let Q(x) be the quotient. & R is the remainder.

by division:
dividend = (divisor × quotient) + Remainder

$P(x) = (x-b)Q(x) + R$

$P(b) = (b-b)Q(b) + R$

$P(b) = (0)Q(b) + R$

$P(b) = R$

Example 3: Apply and Verify the Remainder Theorem

(a) Use the remainder theorem to determine the remainder when

$P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$

$P(-1) = 2(-1)^3 + (-1)^2 - 3(-1) - 6$
$= -2 + 1 + 3 - 6$
$= -4$

$\therefore R = -4$

(b) Verify using long division

$$
\begin{array}{r}
2x^2 - x - 2 \\
x+1 \overline{)\, 2x^3 + x^2 - 3x - 6} \\
\underline{2x^3 + 2x^2 \quad\quad\quad} \\
-x^2 - 3x \quad \\
\underline{-x^2 - x \quad\;} \\
-2x - 6 \\
\underline{-2x - 2} \\
-4
\end{array}
$$

$\therefore R = -4$

(c) Use the remainder theorem to determine the remainder when

$P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$

$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 6$

$= 2\left(\frac{27}{8}\right) + \frac{9}{4} - \frac{9}{2} - 6$

$= \frac{27}{4} + \frac{9}{4} - \frac{18}{4} - \frac{24}{4}$

$= \frac{-6}{4}$

$= -3/2$

$\therefore R = \frac{-3}{2}$

Example 4: Solve for an Unknown Coefficient

Determine the value of *k* such that when $3x^4 + kx^3 - 7x - 10$ is divided by $x - 2$, the remainder is 8

$P(2) = 8$

$3(2)^4 + k(2)^3 - 7(2) - 10 = 8$

$48 + 8k - 14 - 10 = 8$

$24 + 8k = 8$

$8k = 8 - 24$

$\frac{8k}{8} = \frac{-16}{8}$

$\boxed{k = -2}$

Extend: Synthec Division. A different method for dividing polynomials

Divide $x^3 + 7x^2 - 3x + 4$ *by* $x + 2$

$Q(x) = x^2 + 5x - 13$

$R = 30$

$$\begin{array}{r|rrrr} -2 & 1 & 7 & -3 & 4 \\ & & -2 & -10 & +26 \\ \hline & 1 & 5 & -13 & 30 \end{array}$$

example 4 Redux:

$3x^4 + kx^3 - 7x - 10 \div x - 2$ $\quad R = 8$

$$\begin{array}{r|rrrrr} 2 & 3 & k & 0 & -7 & -10 \\ & & 6 & 12+2k & 24+4k & 8k+34 \\ \hline & 3 & 6+k & 12+2k & 4k+17 & 8k+24 \end{array}$$

$8k + 24 = 8$

$8k = 8 - 24$

$8k = -16$

$k = -2$

Homework:

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