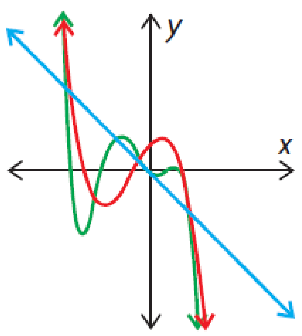
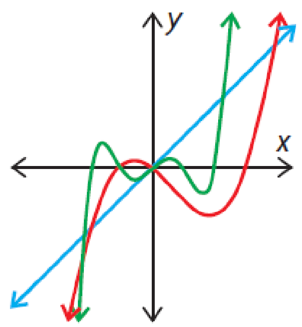
Review of Key Concepts: 1.2 Characteristics of Polynomial Functions

- Polynomial functions of the same degree have similar characteristics
- The degree and leading coefficient of the equation of the polynomial function indicate the end behaviour of the graph
- The degree of a polynomial function provides information about the shape, turning points, and zeros of the graph.

Polynomial Functions of Odd Degree

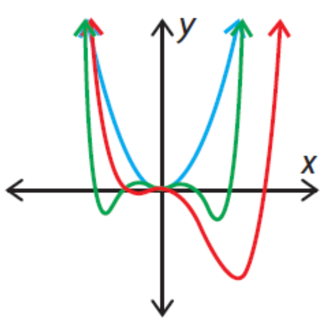


Positive Leading Coefficient

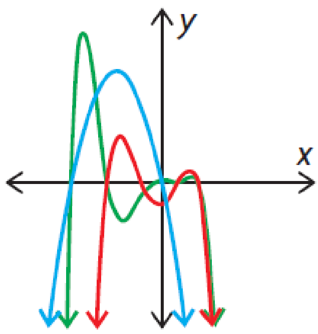


Negative Leading Coefficient

Polynomial Functions of Even Degree



Positive Leading Coefficient



Negative Leading Coefficient

Turning Points

• A polynomial function of degree *n* has at most *n - 1* turning points.

Number of Zeros

• A polynomial function of degree *n* may have up to *n* distinct zeros.

• A polynomial function of odd degree must have at least one zero.

• A polynomial function of even degree may have no zeros.

Symmetry

• Odd-degree polynomials may have point symmetry

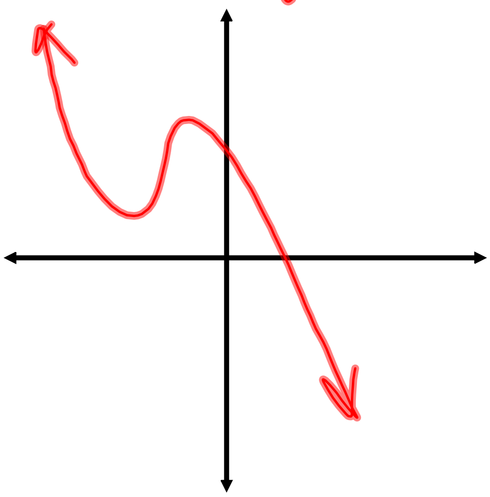
• Even-degree polynomials may have line symmetry

Further Example Problems:

1) Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function.

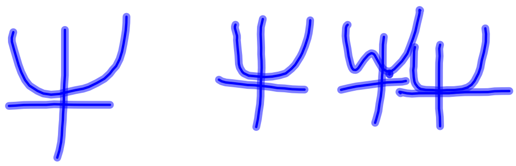
a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

End Behavior:
$x \to \infty;\ y \to -\infty$
$x \to -\infty,\ y \to \infty$



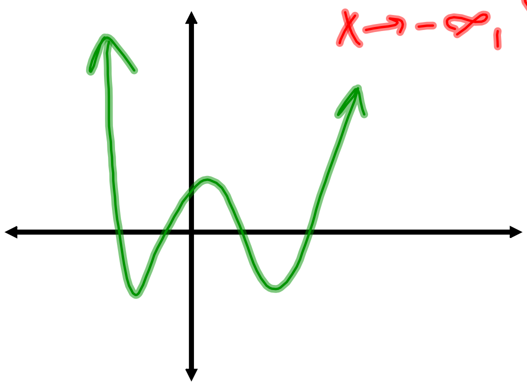
Possible # of turning pts:
2, 4

Possible # of zeros:
1, 2, 3, 4



b) $f(x) = 2x^4 + x^2 + 2$

End Behaviour: $x \to \infty,\ y \to \infty$
$x \to -\infty,\ y \to \infty$

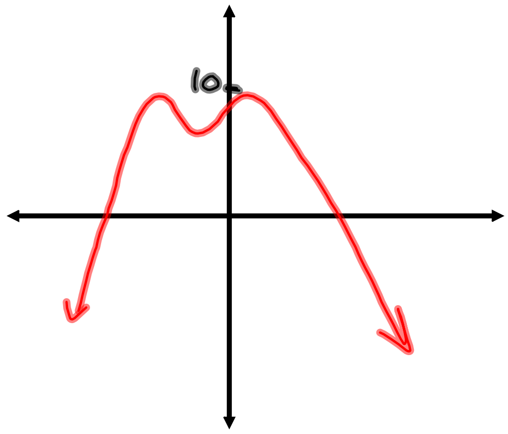


Possible # of turning pts
1, 3

Possible # of zeros:
0, 1, 2, 3

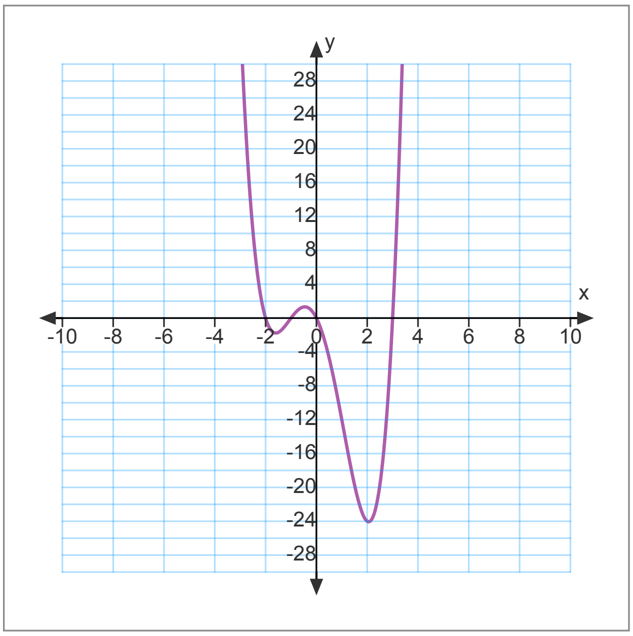
2) What could the graph of a polynomial function that has range $\{y \in \Re \mid y \leq 10\}$ and 3 turning points look like? What can you conclude about its equation?

- if range is $\{y \in \mathbb{R} \mid y \leq 10\}$, function must have end behavior of $x \to \infty, y \to -\infty$; $x \to -\infty, y \to -\infty$
- 3 turning pts $\longrightarrow$ degree = 4!



1.3: Equations and Graphs of Polynomial Functions

Graph: $y = x(x-3)(x+2)(x+1)$



Determine:

a) degree → 4

b) sign of leading co-efficient "+"

c) x - intercepts -2, 0, 3

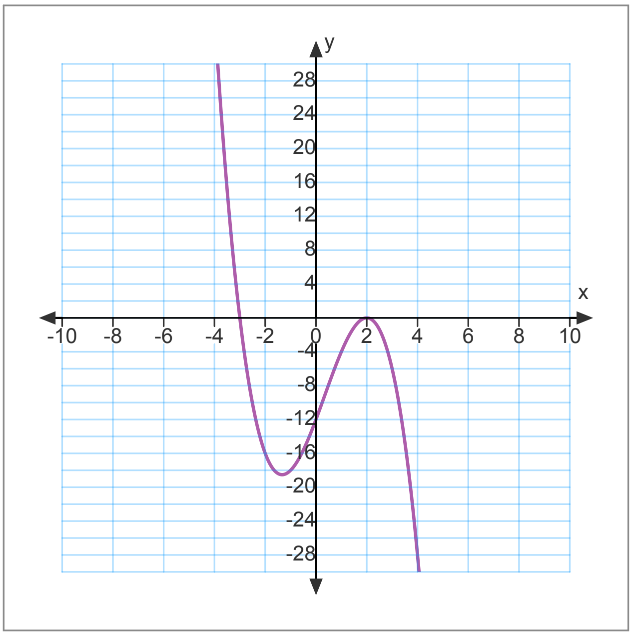
d) y - intercept → 0

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1,0)$	$(0,3)$	$(3,\infty)$
Sign of *f(x)*	+	−	+	−	+

Reflect:

- How can you determine the degree and the sign of the leading coefficient from the equation?
- What is the relationship between the x-intercepts and the equation of the function? the y-intercept and the equation of function?
- What happens to the sign of *f(x)* near each x-intercept?

Graph: $y = -(x-2)^2(x+3)$



Determine:

a) degree $\longrightarrow$ 3

b) sign of leading co-efficient "−"

c) x - intercepts −3 and 2

d) y - intercept $\rightarrow$ −12

Interval	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$		
Sign of *f(x)*	+	−	−		

Graph: $y=(x-2)^3$

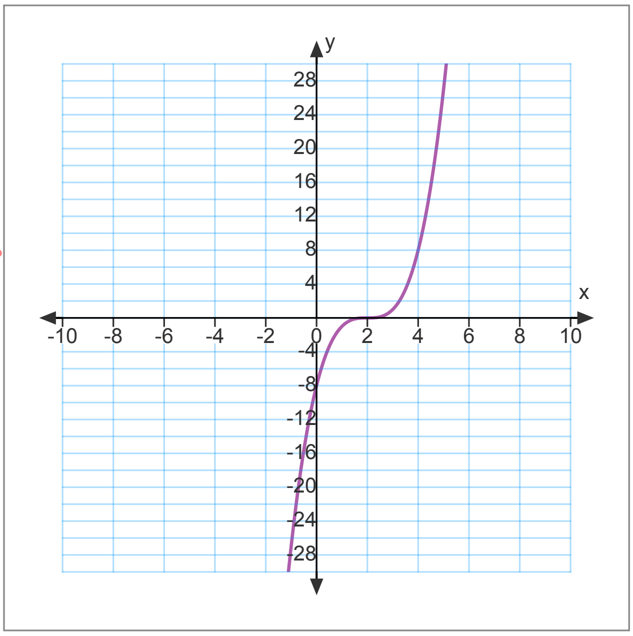
Determine:

a) degree → 3

b) sign of leading co-efficient "+"

c) x - intercepts → 2

d) y - intercept → -8



Interval	$(-\infty,2)$	$(2,\infty)$			
Sign of *f(x)*	−	+			

The Zeros of a Polynomial Function

- The zeros of a polynomial function $y = f(x)$ correspond to the x-intercepts of the graph and to the roots of the corresponding equation $f(x) = 0$.

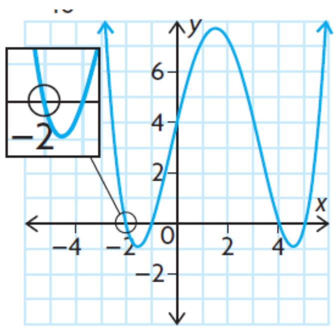
- If a polynomial function has a factor $(x - a)$ that is repeated n times, then $x = a$ is a zero of **order** n.

- The graph of a polynomial function **changes sign** (from positive to negative or negative to positive) at zeros that are of **odd** order but does **not change sign** at zeros that are of **even** order

- linear factors of a polynomial function:

Curve passes through x - intercept

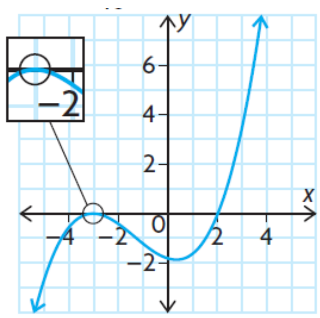
The graph has a linear shape near this *x*-intercept.



- squared factors of a polynomial function:

x-intercepts are turning points

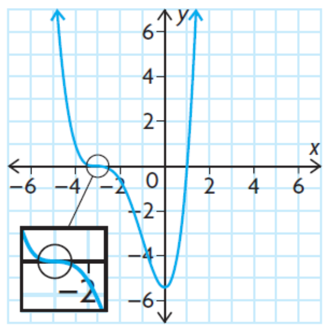
graph has a parabolic shape near these x - intercepts



- cubed factors of a polynomial function:

curve passes through x-intercepts and x-axis is tangent to curve

graph has a cubic shape near these x - intercepts



Example 1

For each graph of a polynomial function, determine:

i) the least possible degree and the sign of the leading coefficient

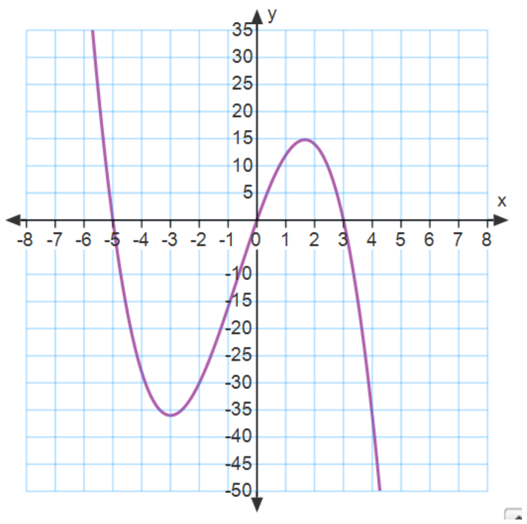
ii) the x-intercepts and the factors of the function

iii) the intervals where the function is positive and the intervals where the function is negative

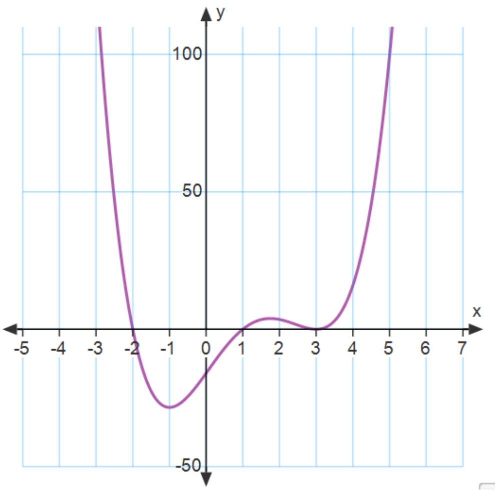
(i) least possible degree = 3

(ii) x-int's are: −5, 0, and 3

factors: $x(x+5)(x-3)$



Interval	$(-\infty, -5)$	$(-5, 0)$	$(0, 3)$	$(3, \infty)$
sign of *f(x)*	+	−	+	−



(i) least possible degree = 4

(ii) x-int's: $-2, 1, 3$

factors: $(x+2)(x-1)(x-3)^2$

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
sign of *f(x)*	+	−	+	+

Example 2

Sketch a graph of each polynomial function:

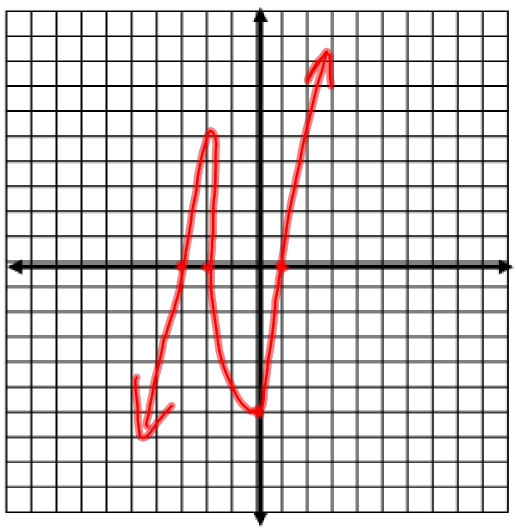
a) $y=(x-1)(x+2)(x+3)$

b) $y=-2(x-1)^2(x+2)$

c) $y=-(2x+1)^3(x-3)$

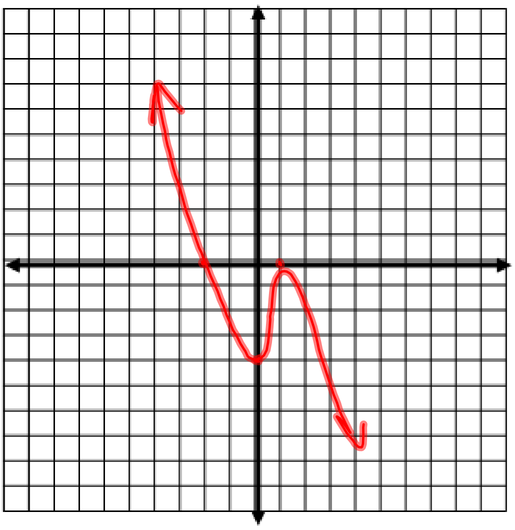
(a)

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
3	"+"	$x \to \infty$ $y \to \infty$ $x \to -\infty$ $y \to -\infty$	1, -2, -3 all order 1	-6



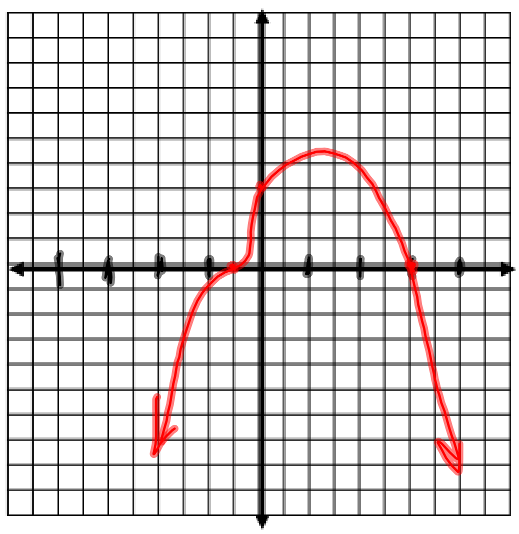
(b) $y = -2(x-1)^2(x+2)$

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
3	" – "	$x \to \infty$, $y \to -\infty$; $x \to -\infty$, $y \to \infty$	1 (order 2); -2 (order 1)	-4



(c) $y = -(2x+1)^3(x-3)$

Degree	Leading Coefficient	End Behaviour	Zeros and x-intercepts	y-intercept
4	" $-$ "	$x \to \infty$ $y \to -\infty$ $x \to -\infty$ $y \to -\infty$	$-1/2$ (order 3) 3 (order 1)	3



Example 3

Write the equation of a cubic function that has zeros at -2, 3 and $\frac{2}{5}$

The function also has a *y*-intercept of 6.

$$y = a(x+2)(x-3)(2x-5)$$

sub pt (0,6)

$$6 = a(0+2)(0-3)(2(0)-5)$$

$$6 = a(2)(-3)(-5)$$

$$\frac{6}{30} = \frac{\cancel{30}a}{\cancel{30}}$$

$$a = \frac{1}{5} \quad \therefore y = \frac{1}{5}(x+2)(x-3)(2x-5)$$

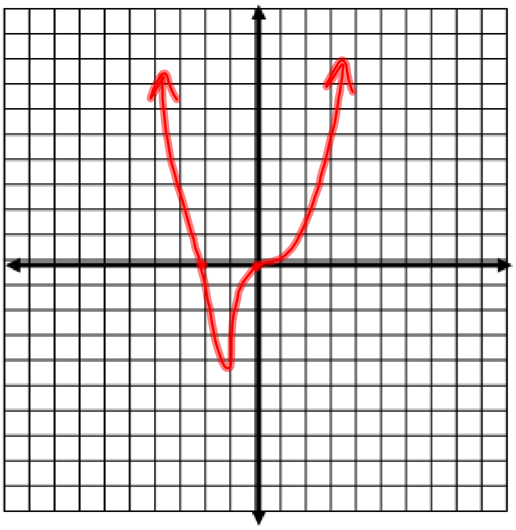
Example 4

Sketch the graph of $f(x) = x^4 + 2x^3$

- factor 1st to find zeros

$f(x) = x^3(x+2)$

Zeros: 0 and -2



Even and Odd Degree Functions

Even Functions

An even-degree polynomial function is an **even function** if the exponent of each term of the equation is even. An even function satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$. An even function is symmetric about the y-axis

Odd Functions

An odd-degree polynomial function is an odd **function** if each term of the equation has an odd exponent. An odd function satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$. An odd function is rotationally symmetric about the origin.

Example 5

Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Verify your response.

a) $f(x) = 2x^4 - 5x^2 + 4$

b) $f(x) = -3x^5 + 9x^3 + 2x$

c) $f(x) = 2x(x+1)(x-2)$

d) $f(x) = x^6 - 4x^3 + 6x^2 - 4$

(a) $f(-x) = 2(-x)^4 - 5(-x)^2 + 4$

$f(-x) = 2x^4 - 5x^2 + 4$

$\therefore f(x) = f(-x)$

$\therefore f(x)$ is even

(b) $f(-x) = -3(-x)^5 + 9(-x)^3 + 2(-x)$

$f(-x) = 3x^5 - 9x^3 - 2x$

$f(-x) = -1(-3x^5 + 9x^2 + 2x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is odd

(c) $f(-x) = 2(-x)(-x+1)(-x-2)$

$f(-x) = -2x(-1)(x-1)(-1)(x+2)$

$f(-x) = -2x(x-1)(x+2)$

$-f(x) = -1[2x(x+1)(x-2)]$

$-f(x) = -2x(x+1)(x-2)$

$\therefore f(x) \neq f(-x) \neq -f(x)$

$\therefore f(x)$ is niether even nor odd

$f(x) = x^6 - 4x^3 + 6x^2 - 4$

$f(-x) = (-x)^6 - 4(-x)^3 + 6(-x)^2 - 4$

$f(-x) = x^6 + 4x^3 + 6x - 4$

$-f(x) = -1(x^6 - 4x^3 + 6x - 4)$

$= -x^6 + 4x^3 - 6x + 4$

$\therefore f(x) \neq f(-x) \neq -f(x)$

$\therefore f(x)$ is niether even nor odd.